ADDENDUM: THE CASE OF CLOSED SURFACES. (BOUNDARY VALUE PROBLEMS ON PLANAR GRAPHS AND FLAT SURFACES WITH INTEGER CONE SINGULARITIES, I: THE DIRICHLET PROBLEM)

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Abstract. We extend our discrete uniformization theorems for planar, m-connected, Jordan domains [Journal für die reine und angewandte Mathematik 670 (2012), 65–92] to closed surfaces of genus $m \geq 1$.

0. Introduction

The aim of this addendum to our paper [1] is to extend the results therein to the case of closed, triangulated surfaces, of any non-positive genus. The new ingredient we employ is the *Hauptvermutung* for 2-manifolds with boundary. The extension provided here is natural and straightforward. In fact, Corollary 0.3 in the above mentioned paper already provided a discrete unifomization theorem for closed surfaces; those that are obtained by doubling an m-connected ($m > 1$), planar, Jordan domain along its full boundary.

By a singular flat surface, we will mean a surface which carries a metric structure locally modeled on the Euclidean plane, except at a finite number of points. These points have cone singularities, and the cone angle is (in general) allowed to take any positive value (see for instance [4] for a detailed survey). Following the convention in [1], a Euclidean rectangle will denote the image under an isometry of a planar Euclidean rectangle, and a singular flat, genus zero compact surface with $m \geq 3$ boundary components, will be called a *ladder of singular pairs of pants*.

In order to make the statement of the theorem from [1] that will be used here self-contained, recall that in [1] we considered a planar, bounded, m-connected region Ω, where $\partial\Omega$ denotes its piecewise linear boundary. Let $\mathcal{T}$ be a triangulation of $\Omega \cup \partial\Omega$. Let $\partial\Omega = E_1 \sqcup E_2$, where E_1 is the outermost component of $\partial\Omega$. We invoke a *conductance function* on $\mathcal{T}^{(1)}$, making it a *finite network*, and use it to define a *combintorial Laplacian* Δ on $\mathcal{T}^{(0)}$.

For any positive constant k, let g be the solution of a *discrete Dirichlet boundary value problem* defined on $\mathcal{T}^{(0)}$ which is determined by requiring that $g|_{\mathcal{T}^{(0)}\cap E_1} = k, g|_{\mathcal{T}^{(0)}\cap E_2} = 0$, and that $\Delta g = 0$ at every interior vertex of $\mathcal{T}^{(0)}$. Furthermore, let $E(g)$ be the *Dirichlet energy* of g, and let $\frac{\partial g}{\partial n}(x)$ denote the *normal derivative* of g at the vertex $x \in \partial\Omega$ (cf. [1, Section 1] for further details).

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We now recall one of the main results of [1].

Theorem 0.1. [1, Theorem 0.2] *Let $(\Omega, \partial\Omega, \mathcal{T})$ be a bounded, m-connected, planar, Jordan region with $E_2 = E_2^1 \sqcup E_2^2 \ldots \sqcup E_2^{m-1}$. Let S_Ω be a ladder of singular pairs of pants such that*

(1) $\mathrm{Length}_{\mathrm{Euclidean}}(S_\Omega)_{E_1} = \sum_{x \in E_1} \frac{\partial g}{\partial n}(x)$, *and*
(2) $\mathrm{Length}_{\mathrm{Euclidean}}(S_\Omega)_{E_2^i} = -\sum_{x \in E_2^i} \frac{\partial g}{\partial n}(x)$, *for* $i = 1, \cdots, m-1$,

where $(S_\Omega)_{E_1}$ and $(S_\Omega)_{E_2^i}$, for $i = 1, \cdots, m-1$, are the boundary components of S_Ω. Then, there exists a mapping f which associates to each edge in $\mathcal{T}^{(1)}$ a unique Euclidean rectangle in S_Ω in such a way that the collection of these rectangles forms a tiling of S_Ω. Furthermore, f is boundary preserving, and f is energy preserving in the sense that $E(g) = \mathrm{Area}(S_\Omega)$.

In the statement of the theorem, "boundary preserving" means that the rectangle associated to an edge $[u, v]$ with $u \in \partial\Omega$ has one of its edges on a corresponding boundary component of the singular surface.

Given a domain as in the theorem, one may look at the *closed surface* obtained by doubling the domain along its full boundary. The following Corollary provides discrete uniformization for this class of surfaces.

Corollary 0.2. [1, Corollary 0.3] *Under the assumptions of Theorem 0.1, there exists a canonical pair (S, f), where S is a flat surface with conical singularities of genus $(m-1)$ tiled by Euclidean rectangles, and f is an energy preserving mapping from $\mathcal{T}^{(1)}$ into S, in the sense that $2E(g) = \mathrm{Area}(S)$. Moreover, S admits a pair of pants decomposition whose dividing curves have Euclidean lengths given by $(1)-(3)$ of Theorem 0.1.*

Proof. Given $(\Omega, \partial\Omega, \mathcal{T})$, glue together two copies of S_Ω (their existence is guaranteed by Theorem 0.1) along corresponding boundary components. This results in a flat surface $S = S_\Omega \bigcup_{\partial\Omega} S_\Omega$ of genus $(m-1)$ and a mapping $\bar{f}$ which restricts to f on each copy.

□

We are now ready to address the case of an arbitrary closed surface of genus $m \geq 1$, the main purpose of this addendum. Henceforth, we will use the following standard notation. If K is a complex, then $|K|$ denotes the union of the elements in K, endowed with the subspace topology induced by the topology on R^3.

Theorem 0.3 (Discrete Uniformization of Closed Surfaces)**.** *Let $(X, \mathcal{T})$ be a closed, triangulated surface of genus m, $m \geq 1$. Then there exists a mapping f which associates to each edge in a refined triangulation of $\mathcal{T}^{(1)}$ a unique Euclidean rectangle in a singular flat surface which is homeomorphic to X and denoted by S. Each one of the cone singularities in S is an integer multiple of π. Moreover, the collection of these rectangles forms a tiling of S. Finally, S admits a pair of pants decomposition whose dividing curves have Euclidean lengths given by $(1)-(3)$ of Theorem 0.1.*

Proof. We first assume that X, considered now as a simplicial complex, is a surface with negative Euler characteristic $\chi(X) = 2 - 2m$. Let γ be a disjoint union of $m+1$ embedded, closed, 1-cycles in $\mathcal{T}^{(1)}$ such that if $(S, \mathcal{T})$ is cut along γ, the resulting pieces $(P_1, \mathcal{T}_1)$ and $(P_1, \mathcal{T}_1)$, are triangulated, genus 0, pair of pants, each of which having $m+1$ boundary

components; where $\mathcal{T}_1 = \mathcal{T}|P_1$ and $\mathcal{T}_2 = \mathcal{T}|P_2$. Note that the boundary components of $(P_1, \mathcal{T}_1)$ and of $(P_1, \mathcal{T}_1)$ are in one to one correspondence with the components of γ.

The classification of 2-manifolds with boundary (see for instance [2, section 5]) implies that $|P_1|$ and $|P_2|$ are homeomorphic. Hence, it follows from the Hauptvermutung in dimension 2 (see for instance [2, 3]) that P_1 and P_2 are combinatorially equivalent. Henceforth, let L_1 be a subdivision of $\mathcal{T}_1$ and L_2 a subdivision of $\mathcal{T}_2$, such that (P_1, L_1) and (P_2, L_2) are combinatorially isomorphic.

Let us choose the same conductance constants, $\mathcal{C}$, for L_1 and L_2. From a topological perspective, only the *planarity* of the domain Ω was used in the proof of Theorem 0.1. Hence, we may apply Theorem 0.1 to the (isomorphic) networks (P_1, L_1) and (P_2, L_2) respectively. Once the conductance constants are given, the assertions of Theorem 0.1 are determined solely by the combinatorial isomorphism class of the triangulation. Therefore, it follows that the two ladder of singular pair of pants S_{P_1} and S_{P_2}, whose existence is guaranteed by Theorem 0.1, are in fact identical.

Therefore, in this case the assertions of Theorem 0.3 now follow by setting S to be the singular flat surface obtained by gluing the two isometric singular pair of pants S_{P_1} and S_{P_2} along matching boundary components. Also, [1, Equation (4.12)] justifies the assertion regarding the cone angles.

We now treat the case of $m = 1$, i.e., $(X, \mathcal{T})$ is a triangulated torus. Choose in $\mathcal{T}^{(1)}$ an embedded, piecewise-linear, 1-cycle, τ, that is a meridian. Hence, when $(X, \mathcal{T})$ is cut along τ, the result is a triangulated cylinder denoted by S_τ. Let τ_1 and τ_2 be the two boundary components of S_τ. Theorem 0.4 in [1] provides a discrete uniformization in the case of a triangulated annulus. Since only the planarity of the annulus was used in the proof, we may apply it to the case of a triangulated cylinder. In particular, the proof shows that the image of the cylinder S_τ under the mapping f is a straight Euclidean cylinder whose height equals k and whose circumference equals

$$\sum_{x \in \tau_1} \frac{\partial g}{\partial n}(x).$$

Hence, in this case the assertions of Theorem 0.3 follow by letting S be the flat torus obtained by gluing the top and the bottom of this Euclidean cylinder thus obtained, by an isometry. □

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